FS Exercises

Ex. 1

$$x(t) = 10 + 3\cos w_0 t + 5\cos(2w_0 t + 30)$$

Find ① Average value X_0

② $X_1, X_{-1}, X_2, X_{-2}, X_3, X_{-3}$

③ Find the TFS coefficients a_0, a_1, a_2, b_1, b_2

④ Find the average power

Sol.

① $X_0 = 10$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jkw_0}$$

$$x(t) = 10 + \frac{3}{2}\left[e^{jw_0 t} + e^{-jw_0 t}\right] + \cancel{\frac{5}{2}\left[e^{jw_0 2t} + e^{-jw_0 2 t}\right]}$$

$$+ \frac{5}{2}\left[e^{j(2w_0 t+30)} + e^{-j(2w_0 t+30)}\right]$$

$$= 10 + \underset{\overset{\uparrow}{X_1}}{\frac{3}{2}} e^{j1.w_0 t} + \underset{\overset{\uparrow}{X_{-1}}}{\frac{3}{2}} e^{-j1.w_0 t} + \underbrace{\frac{5}{2} e^{j\frac{\pi}{6}}}_{X_2} e^{j2.w_0 t} + \underbrace{\frac{5}{2} e^{-j\frac{\pi}{6}}}_{X_{-2}} e^{-j2w_0 t}$$

② $X_1 = \frac{3}{2}$ $\qquad X_{-1} = 3/2$

$X_2 = \frac{5}{2} \angle \pi/6$ $\qquad X_{-2} = \frac{5}{2} \angle -\pi/6$

$X_3 = X_4 = X_5 = \cdots = 0$

③ $a_n = 2\,Re[X_n] \Rightarrow a_0 = X_0$ $\qquad a_2 = \frac{5}{2}\cos\frac{\pi}{6}$

$a_1 = 3/2$ $\qquad a_3 = 0$

$b_n = -2\,Im[X_n]$ $\qquad b_1 = 0$

$b_2 = -2 * Im\left[\frac{5}{2}\cos\frac{\pi}{6} - j\frac{5}{2}\sin\frac{\pi}{6}\right]$

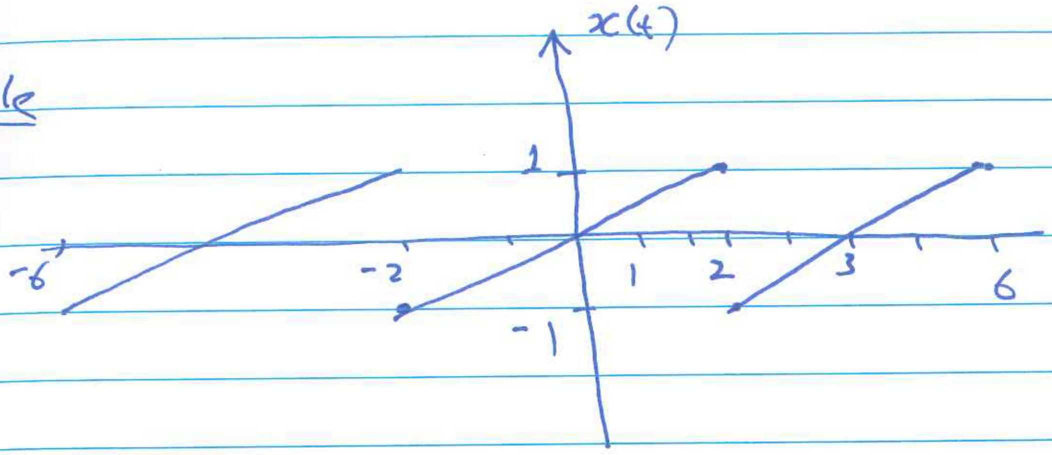
$= 5 \sin \pi/6$

④ $$P_x = |X_0|^2 + \sum_{n=1}^{\infty} 2|X_n|^2$$

$$= (10)^2 + 2\left(\frac{3}{2}\right)^2 + 2\left(\frac{5}{2}\right)^2$$

$$= 100 + 2\left[\frac{9}{4} + \frac{25}{4}\right] = 117 \text{ W}$$

Example



find X_n : Complex Exp. FS coefficients

Sol.

$$X_0 = \frac{1}{T_0}\int_{\langle T_0\rangle} x(t)\,dt = 0$$ (Average value or DC value)

$$T_0 = 4 \quad , \quad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \frac{\pi}{2}.$$

$$X_n = \frac{1}{T_0}\int_{\langle T_0\rangle} x(t)\,e^{-jn\omega_0 t}\,dt$$

$$x(t) = \frac{t}{2} \quad , \quad -2 < t < 2$$

$$\Rightarrow X_n = \frac{1}{4}\int_{-2}^{2} \frac{t}{2}\,e^{-j\frac{\pi}{2}nt}\,dt$$

$$= \frac{1}{8}\int_{-2}^{2} t\,e^{-j\frac{\pi}{2}nt}$$

$$\vdots$$

$$= \frac{j}{\cancel{2}\pi n}\left[\cos \pi n\right] \cancel{-\frac{j}{\pi^2 n^2}\overset{0}{\sin \pi n}}$$

$$= \frac{j}{n\pi}(-1) = \frac{-j}{n\pi}$$

Integration by parts

$$\begin{array}{ccc} t & & e^{-j\frac{\pi}{2}nt} \\ & \searrow^{+} & \\ 1 & & \frac{1}{-j\frac{\pi}{2}n}\,e^{-j\frac{\pi}{2}nt} \\ & \searrow^{-} & \\ 0 & & \frac{-1}{\frac{\pi^2}{4}n^2}\,e^{-j\frac{\pi}{2}nt} \end{array}$$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{-j}{n\pi}\right) e^{jn\frac{\pi}{2}t}$$

$X_0 = 0$

$X_1 = \frac{-j}{\pi} = \frac{1}{\pi} \angle -90$ $\qquad X_{-1} = \frac{1}{\pi} \angle 90$

$X_2 = \frac{-j}{2\pi} = \frac{1}{2\pi} \angle -90$ $\qquad X_{-2} = \frac{1}{2\pi} \angle 90$

$\vdots$

Example $x(t) = 1 + \sin 6000\pi t \cos^2 8000\pi t$

Ⓐ Find X_n ? Ⓑ plot X_n's.

Ⓒ Find P_{av} ?

$x(t)$ can be directly expanded in sin/cos. no need for integration

$$x(t) = 1 + \left(\frac{e^{j6000\pi t} + e^{-j6000\pi t}}{j2}\right)\left(\frac{e^{j8000\pi t} + e^{-j8000\pi t}}{2}\right)^2$$

$$= 1 - \frac{1}{j8} e^{-j22000\pi t} + \frac{1}{j8} e^{j22000\pi t}$$

$$+ \frac{1}{j8} e^{-j10000\pi t} - \frac{1}{j8} e^{j10000\pi t}$$

$$- \frac{2}{j8} e^{-j6800\pi t} + \frac{2}{j8} e^{j6000\pi t}$$

Freq: $\left.\begin{matrix} 22000\pi \\ 10000\pi \\ 6000\pi \end{matrix}\right\} \Rightarrow w_0 = 2000\pi$

$T_{01} = \frac{2\pi}{w_{01}} = \frac{1}{11000}$ $T_{02} = \frac{1}{5000}$ $T_{03} = \frac{1}{3000}$

$\frac{T_{01}}{T_{02}} = \frac{5}{11}$ $\frac{T_{01}}{T_{03}} = \frac{3}{11}$

Find $LCM(11, 11) = 11 \Rightarrow T_0 = 11 * T_{01} = \frac{1}{1000}$

$\Rightarrow w_0 = \frac{2\pi}{T_0} = 2000\pi$

$22000\pi \longrightarrow n = \frac{22000}{2000} = 11$

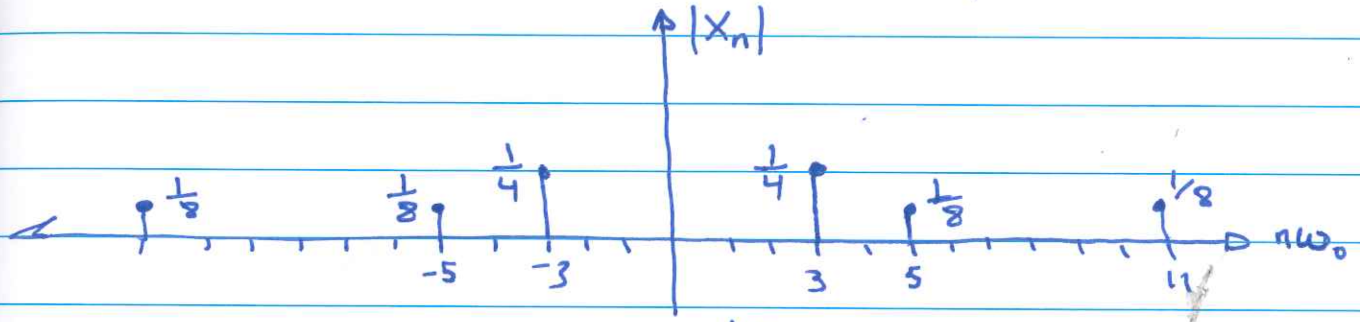
$10\,000\,\pi \longrightarrow n = 5$

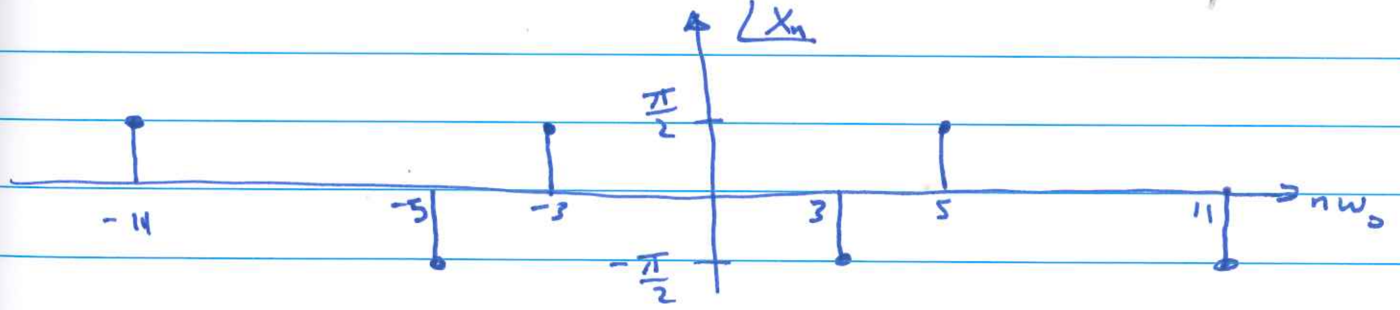
$6000\,\pi \longrightarrow n = 3$

$\Rightarrow X_0 = 1 \qquad X_{11} = \frac{1}{8j} = \frac{1}{8}\angle -90 \quad \Big| \quad X_{-11} = -\frac{1}{8j} = X_{11}^*$

$X_5 = -\frac{1}{8j} = \frac{1}{8}\angle 90 \quad \Big| \quad X_{-5} = +\frac{1}{8j} = X_5^*$

$X_3 = \frac{2}{8j} = \frac{2}{8}\angle -90 \quad \Big| \quad X_{-3} = -\frac{2}{8j}$



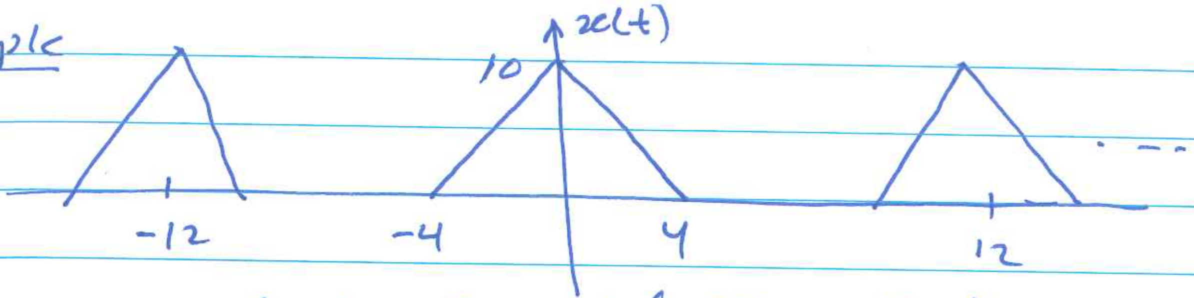


ⓒ Power in $x(t) = |X_0|^2 + \sum_{n=1}^{\infty} 2|X_n|^2$

$= 1 + 2\cdot\frac{1}{16} + 2\cdot\frac{1}{64} + 2\cdot\frac{1}{64}$

$= \frac{19}{16}$ W

Example



A) Determine Complex Exponential FS coefficients ? X_n

B) = TFS coefficients ? a_n, b_n

C) = Compact TFS coefficients ? D_n

Sol. $T_0 = 12 \Rightarrow w_0 = \frac{2\pi}{T_0} = \frac{\pi}{6}$

$x(t)$ is even $\Rightarrow \boxed{b_n = 0}$ $n = 1, 2, \ldots$

$$a_0 = \frac{1}{T_0}\int_{-4}^{4} x(t)\,dt = \frac{1}{12}\cdot \text{Area} = \frac{1}{12}\cdot 4 . 10 = \frac{10}{3}$$

$$a_n = \frac{2}{T_0}\int_{-4}^{4} x(t)\cos nw_0t\,dt$$

$$x(t) = \begin{cases} \frac{5}{2}(t+4) & -4 < t < 0 \\ -\frac{5}{2}(t-4) & 0 < t < 4 \end{cases}$$

$$a_n = \frac{2}{12}\int_{-4}^{0} \frac{5}{2}(t+4)\cos nw_0t + \frac{2}{12}\int_{0}^{-4} -\frac{5}{2}(t-4)\cos nw_0t$$

$$= \frac{5}{12}\int_{-4}^{0} t\cos nw_0t + \frac{5.4}{12}\int_{-4}^{0}\cos nw_0t + -\frac{5}{12}\int_{0}^{4} t\cos nw_0t + \frac{5.4}{12}$$

$$+ \frac{5.4}{12}\int_{0}^{-4}\cos nw_0t$$

$$= \frac{5}{12}\left[\frac{t}{nw_0}\sin nw_0t + \frac{1}{(nw_0)^2}\cos nw_0t\right|_{-4}^{0} + \frac{5}{3}\frac{1}{nw_0}\sin nw_0t\Big|_{-4}^{0}$$

$$-\frac{5}{12}\left[\frac{t}{nw_0}\sin nw_0t + \frac{1}{(nw_0)^2}\cos nw_0t\right|_{0}^{4} + \frac{5}{3}\frac{1}{nw_0}\sin nw_0t\Big|_{0}^{4}$$

$$= \cancel{2 * \frac{5}{3} * \frac{1}{n w_0} \sin\left(\frac{2}{3}\pi n\right)} + \frac{5}{12}\Big[2 * \frac{1}{(n w_0)^2} - \cancel{2 * \frac{4}{n w_0} \sin \frac{2}{3}\pi n}$$

$$- 2 * \frac{1}{(n w_0)^2} \cos\left(\frac{2}{3}\pi n\right)\Big]$$

$$= \cancel{\frac{5}{12}\Big[2 *}$$

$$\frac{5}{12} \cdot 2 \cdot \frac{1}{(n w_0)^2}\left[1 - \cos \frac{2}{3}\pi n \right]$$

$$a_n = \frac{30}{n^2 \pi^2}\left[1 - \cos\left(\frac{2}{3}\pi n\right)\right] \qquad n = 1, 2, 3, \ldots$$

Complex FS :

$$X_n = \frac{1}{2}(a_n - j b_n) = \frac{1}{2} a_n = \frac{15}{n^2 \pi^2}\left(1 - \cos \frac{2}{3}\pi n\right)$$

$$X_0 = a_0 = \frac{10}{3} \qquad n = 1, 2, \ldots$$

Compact TFS :

$$x(t) = X_0 + \sum \underbrace{2|X_n|}_{D_n} \cos(n w_0 t + \theta_n)$$

$$D_n = 2 * \frac{15}{n^2 \pi^2}\left(1 - \cos \frac{2}{3}\pi n\right)$$

$$\theta_n = 0$$

$$X_0 = \frac{10}{3}$$